

# EXPLICIT MASS RENORMALIZATION AND CONSISTENT DERIVATION OF RADIATIVE RESPONSE OF CLASSICAL ELECTRON

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**ABSTRACT.** The radiative response of a classical electron is commonly described by Lorentz-Abraham-Dirac (LAD) equation. In Dirac's derivation of this equation Poincaré stresses are ignored, though non-electromagnetic cohesive forces are implicitly taken into account through the mass renormalization. Our approach to the mass renormalization is to introduce Poincaré cohesive forces explicitly through "adjacent potentials" in the relativistic covariant Lagrangian formulation of the problem. We derive the relativistic point charge dynamics from an extended charge dynamics in a localization limit. We consistently derive LAD equation by a method which can be viewed as a refinement of Dirac's approach in the spirit of Ehrenfest theorem. The model exhibits the mass renormalization as the cancellation of Coulomb energy with the Poincaré cohesive energy. The resulting LAD equation involves a coefficient  $\vartheta$  at the radiation response term, the value of the coefficient depends on the composition of Poincaré forces. The coefficient may be negative, forbidding runaway solutions.

## 1. INTRODUCTION

The dynamics of a classical electron which involves its radiative response is described by Lorentz-Abraham-Dirac (LAD) equation which has the form, [9], [14], [16], [21], [22]:

$$(1) \quad m\dot{v}_\mu = ev_\nu F_{\text{ex}\mu}^\nu + \frac{2}{3}e^2\ddot{v}_\mu + \frac{2}{3}e^2(\dot{v}\dot{v})v_\mu.$$

Here the covariant notation is used,  $f_{\text{ex}\mu} = ev_\nu F_{\text{ex}\mu}^\nu$  is the Lorentz force generated by the external field  $F_{\text{ex}}^{\mu\nu}$ ,  $\dot{v}_\mu = \partial_s v_\mu$  and  $(\dot{v}\dot{v})$  is the 4-product in Minkowski space:

$$(2) \quad (vw) = v_\nu w^\nu = g_{\mu\nu}v_\mu w_\nu = v_0w_0 - v_1w_1 - v_2w_2 - v_3w_3 = v_0w_0 - \mathbf{v} \cdot \mathbf{w};$$

we use the units in which the speed of light  $c = 1$  and the summation convention. The radiative response of an electron was originally derived by Abraham and Lorentz from the analysis of Lorentz-Abraham model, see [10], [21], [19], [25], [26]. A relativistic treatment of Lorentz-Abraham model meets with difficulties, [10], [16], [17], [19], [21], [22], [25], [26]. The relativistic derivation of the radiative response based on the analysis of the energy and momentum conservation laws and on mass renormalization is due to Dirac [9] and now is often used, [14], [16], [22].

It is well-known that the derivation of the LAD equation and the equation itself is not without difficulties, [16], [19], [21]. Sometimes the source of the difficulties is attributed to the mass renormalization. The mass renormalization is described

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in [14, Sec. 75] as follows: "When in the equation of motion we write a finite mass for the charge, then in doing this we essentially assign to it a formally infinite negative "intrinsic mass" of nonelectromagnetic origin, which together with the electromagnetic mass should result in a finite mass for the particle. Since, however, the subtraction of one infinity from another is not an entirely correct mathematical operation, this leads to a series of further difficulties". A natural way to deal with the infinities is to introduce an extended charge and then to find its limit dynamics in the localization limit as its size tends to zero, but there are complications [22, Sec. 8.4]: "Unfortunately, it is not easy to obtain a theory in this way that has a local conservation of energy and momentum". *Still our goal here is to obtain such a theory, namely to introduce a classical relativistically covariant model of a charge with fulfillment of the energy and momentum conservation laws which allows asymptotic localization of a charge to a point and is simple enough, so we can find explicitly the limit dynamics and study the effects of the mass renormalization.* The distributed charge is described here not by the Lorentz-Abraham model or its generalizations, [1], [11], [12], [15], [17], [21], [25], but by a relativistic covariant Lagrangian field theory, [5], [6], where the spatial distribution of the charge is not prescribed but rather is determined as a solution of a field equation. To balance destabilizing EM forces we introduce in the model Poincaré cohesive forces of a non-electromagnetic origin explicitly through an additional internal field of a non-electromagnetic nature acting only on the charge itself which we call "adjacent field" of a charge.

In the model which we study here we are able to explicitly observe two infinities – the infinite energy of Coulomb field and the infinite energy of the adjacent field, the infinities emerge as the size of the extended charge tends to zero. Our analysis explicitly shows that these infinite energies cancel out, and only a finite energy responsible for the observable mass remains. Therefore, if Poincaré stresses are properly introduced before passing to the point localization limit, the mass renormalization becomes a self-consistent, explicit and quite regular procedure. In Dirac's derivation of the LAD equation Poincaré stresses are ignored, though the energy and momentum conservation laws are assumed to be fulfilled and implicitly non-electromagnetic cohesive forces are taken into account through the mass renormalization. The relevance of Poincaré stresses for dynamical properties of a charge in the non-radiative case is well-known, [10], [14], [16], [19], [22], [25] and is demonstrated in Poincaré-Schwinger model, [10], [18], [20]. Here we show the impact of Poincaré stresses on the radiative response of an elementary charge (an electron). Our method of determination of the limit dynamics can be considered as a refinement in the spirit of Ehrenfest theorem of Dirac's approach [9]. This method was applied in [5], [6] to derive Einstein's formula in regimes with acceleration in the case where there is no radiative response.

Now we briefly describe basic features of our method. The LAD equation for a point charge motion is derived under the assumption of the asymptotic localization of the energy of the charge. We do not prescribe a mass distribution for a charge, in the process of derivation we obtain Einstein's formula for the equivalence of the mass and energy which is not postulated. The form of the equations of motion is uniquely determined by the original assumptions in a contrast to the original Dirac's derivation which allowed some freedom in the choice of the form of radiative reaction. Remarkably, the equations which we obtain have the form of the classical

LAD equation, but with one important difference. *The radiation reaction term has a constant coefficient  $\theta$  which can take positive or negative values.* The value of the coefficient depends on the choice of Green functions for determination of the EM field and the adjacent field.

Our derivation is based on the following assumptions:

The elementary charge dynamics is described by a Lorentz and gauge invariant Lagrangian which describes the system "charge-fields", involves the following fields and has the following properties:

- (i) Charge distribution field  $\psi$  (it is a complex scalar in the simplest case which we consider here, but not necessary; for example it can be a spinor).
- (ii) Electromagnetic (EM) field which is responsible for the EM interaction between the charges, the EM field is governed by Maxwell's equations.
- (iii) Adjacent field  $A^{\text{ad}}$  of the charge, this field describes an internal interaction of the charge with itself and balances the EM self-interaction of the charge. This field generates Poincaré cohesive forces, it is also responsible for the mass renormalization. The adjacent field is also governed by Maxwell's equations.
- (iv) The sources for the Maxwell equations both for EM field and the adjacent field are the currents  $J^\nu$  which are determined by the charge distribution field  $\psi$ .
- (v) The field equation (a nonlinear Klein-Gordon (KG) equation in the simplest case) for  $\psi$  involves a nonlinearity which provides a localization of the charge distribution characterized by a microscopically small spatial scale  $a$ .

Note that Poincaré cohesive forces are required to balance charge's EM interaction with its own EM field, and we think that the simplest relativistic-invariant way to provide a seamless balance in all regimes is to take a balancing adjacent field which satisfies the same Maxwell's equations as the EM field and enters the "charges-fields" Lagrangian anti-symmetrically with the EM field. Importantly, a Lagrangian-based model allows a freedom in the determination of the solutions of Maxwell's equations for the EM field and the adjacent fields through the choice of Green functions. We use the freedom to study the dependence on the composition of Poincaré forces and assume that the EM potentials are determined in terms of the the current  $J^\nu$  by the formula

$$(3) \quad A^\nu = \vartheta_0 \mathcal{G}_{\text{ret}} J^\nu + (1 - \vartheta_0) \mathcal{G}_{\text{adv}} J^\nu,$$

and the adjacent field potentials by a similar formula

$$(4) \quad A^{\text{ad}\nu} = \vartheta_1 \mathcal{G}_{\text{ret}} J^\nu + (1 - \vartheta_1) \mathcal{G}_{\text{adv}} J^\nu,$$

where  $\mathcal{G}_{\text{ret}}$ ,  $\mathcal{G}_{\text{adv}}$  are solution operators of Maxwell equations in terms of retarded and advanced potentials respectively, and parameters  $\vartheta_0$  and  $\vartheta_1$  may be different.

Note that our derivation of the LAD equation uses only conservation laws for the fields involved and does not use specific information on the Lagrangian and corresponding field equations. We use two structural properties of the Lagrangian: (i) the existence of the symmetric energy-momentum tensor (which can be constructed for relativistic invariant Lagrangians [8], [13], [14]); (ii) the gauge invariance of the Lagrangian which implies fulfillment of the continuity equation and conservation of the total charge for a charge distribution. We derive relativistic point dynamics from the dynamics of distributed fields in the localization limit when the energy and charge densities converge to delta-functions. The main assumption we impose on the fields is that the energy of the system "charge-fields" asymptotically

concentrates at a trajectory  $\hat{\mathbf{r}}(t)$ . The concentration can be described by two microscopic scales: charge size scale  $a$  and intermediate confinement scale  $R$ ; both  $a$  and  $R$  are vanishingly small compared with the macroscopic scale of order 1. The charge distribution  $\psi$  is essentially confined in a ball of radius  $a$  and asymptotically vanish on a sphere  $\partial B_R$  of a larger radius  $R$ , where  $a/R$  is assumed to be vanishingly small. The energy  $E$  of the "charge-fields" system which is confined in the ball  $B_R$  of radius  $R$  converges:  $E \rightarrow E_\infty(t)$  in the localization limit

$$(5) \quad a \rightarrow 0, \quad R \rightarrow 0, \quad \frac{a}{R} \rightarrow 0.$$

We do not assume that in the above limit the quantities  $a, R$  are arbitrary, they may be subjected to additional restrictions, we only need that there exists a sequence  $a_n, R_n$  which satisfies (5). We also assume that on the surface of the sphere  $\partial B_R$  the EM field  $A$  generated by the charge and the adjacent field  $A^{\text{ad}}$  in the regime  $\frac{a}{R} \rightarrow 0$  can be approximated by solutions of the Maxwell equations with point sources located at the trajectory  $\hat{\mathbf{r}}(t)$ . Such assumptions seem to be natural for point-like behavior of the charge, examples of their fulfillment in non-trivial regimes are given in [6], [7]. We do not make any assumptions on the mass of a charge, the Newtonian mass is derived from the equations. An important assumption on the behavior of the external EM field is that it varies significantly only at macroscopic spatial scales, that is at spatial and time scales of order 1 and not at microscopic scales. We want to stress that this assumption is relevant and LAD equation would not describe charge behavior in our model if the external field strongly varies at microscopic scales as, for example, Coulomb field does near its singularity. Namely, our analysis of a distributed charge in Coulomb field as in hydrogen atom, [2], [3], [4], shows its non-classical behavior, in particular discreteness of energy levels. We want to emphasize that the "non-classical" effects emerge in the classical framework and are not based on quantum-mechanical considerations.

**The result of the analysis:**

Under the above assumptions the limit energy  $E_\infty(t)$  must satisfy the relativistic formula

$$(6) \quad E_\infty = M_0 \left(1 - |\partial_t \hat{\mathbf{r}}|^2\right)^{1/2} = M_0 / |\partial_s t|$$

with a constant  $M_0$  which is interpreted as the rest mass of a charge. The 4-trajectory  $z(s) = (t(s), \hat{\mathbf{r}}(t(s)))$  must satisfy a generalized Lorentz-Abraham-Dirac (LAD) equation

$$(7) \quad M_0 \dot{v}_\mu = f_{\text{ex}\mu} + 2\vartheta \left[ \frac{2}{3} q^2 \ddot{v}_\mu + \frac{2}{3} q^2 (\dot{v}\dot{v}) v_\mu \right]$$

where  $v = \partial_s z = \dot{z}$ ,  $\vartheta = \vartheta_0 - \vartheta_1$ ,  $f_{\text{ex}\mu} = q v_\nu F_{\text{ex}\mu}^\nu$  is the Lorentz force generated by the external EM field and  $q$  is the total charge of the charge distribution. The parameters  $q$  and  $M_0$  for the electron take the values  $q = e$  and  $M_0 = m$  as in (1). Note that the equation is completely determined by the assumptions we made. The equation involves the radiative response coefficient  $2\vartheta = 2\vartheta_0 - 2\vartheta_1$  which originates from the composition of Green functions for Maxwell's equations described by (3), (4).

Now we briefly discuss implications of choosing specific values of  $\vartheta_0$  and  $\vartheta_1$ . There are two most interesting choices for  $\vartheta_0$ ,  $\vartheta_0 = 1$  and  $\vartheta_0 = 1/2$ . The first case

$\vartheta_0 = 1$  describes the theory where the EM interaction between all the charges of the system (including itself) is retarded, this case directly agrees with the principle of causality. At the same time the adjunct potential which describes the charge interaction with itself can be a linear combination of the retarded and advanced field and the value of  $\vartheta_1$  is arbitrary. The adjacent potential  $A^{\text{ad}}$  solely describes an internal interaction of an elementary charge such as an electron with itself (see Section 4 for a discussion), therefore we think that causality arguments cannot be applied to such an internal self-interaction and do not impose restrictions on the choice of  $\vartheta_1$ . A time-reversal symmetric choice  $\vartheta_1 = 1/2$  in (4) results in  $\vartheta = 1/2$  and the *generalized LAD equation (7) with  $\vartheta = 1/2$  coincides with the classical LAD equation (1)*. If we set  $\vartheta_1 = -1$ , we obtain in the case of a single charge a symmetry with respect to combined time reversal and EM-adjacent field exchange. In this case  $\vartheta = -2$  and the radiative response is 4 times stronger than in the classical case.

Another possibility is  $\vartheta \leq 0$ . If  $\vartheta_1 = 1$  we obtain  $\vartheta = 0$  and the adjacent field completely balances the EM field in all regimes and there is no EM self-interaction as in Wheeler-Feynman theory and no radiative response for a single charge at all. Note that the observable radiation of the charge does not vanish, it is described as always by Larmor formula, and as always energy and momentum conservation laws are fulfilled. If  $\vartheta_1 > 1$  we obtain a negative value  $\vartheta < 0$  of the radiative response coefficient. For example, we obtain negative  $\vartheta = -1$  in the particular case  $\vartheta_1 = 2$  and the adjacent field can be represented in the form  $A^{\text{ad}\nu} = 2\mathcal{G}_{\text{ret}}J^\nu - \mathcal{G}_{\text{adv}}J^\nu$  and the adjacent field can be interpreted as a superposition of three elementary internal fields: two retarded and one advanced. A negative value of  $\vartheta$  seems to be an admissible alternative. For such negative values of  $\vartheta < 0$  the generalized LAD equation has different properties compared to the classical LAD equation. The runaway solutions of the classical LAD equation turn into rapidly decaying stable solutions and the pre-acceleration does not happen.

Another natural value for  $\vartheta_0$  is  $\vartheta_0 = 1/2$  as in Wheeler-Feinman theory, [23], [24]. If  $\vartheta_1 = 1/2$  too, then  $\vartheta = 0$  and we obtain a theory which at macroscopic scales coincides with Wheeler-Feinman theory without self-interaction and without radiative response for a single charge. The description of the radiative response in this case would require to consider a system of very many interacting charges as in the universal absorber theory. But if we take  $\vartheta_1 = 1/2 - \vartheta$ , we obtain a modified Wheeler-Feinman theory with a non-zero self-interaction in accelerating regimes and a non-zero radiative response for a single charge which does not rely on existence of universal absorber. The value of  $\vartheta$  can be positive or negative. If  $\vartheta_1 = 1$ , that is the adjacent field is retarded, then  $\vartheta = -1/2$  and the pre-acceleration does not happen.

The above discussion shows that a consistent derivation of the radiation response must take into account Poincaré stresses since we have an example where they give a relevant contribution to the macroscopic law of motion of a point charge. A purely field-theoretic analysis without additional data cannot uniquely define the radiative response coefficient and additional experimental data are required for its determination. The experimental methods for verification of LAD equation described in [21], Section 9.3, could be used for the determination of the value of  $\vartheta$ . At the same time our analysis shows that classical electrodynamics is self-consistent

if the concept of an elementary charge is properly introduced, which agrees with conclusions made in [19] on different grounds.

**Conclusions:**

(i) The Lagrangian field model allows to consistently derive the relativistic version of Newton's equations for a localized charge, Einstein's formula and generalized LAD equation.

(ii) The mass renormalization is explicitly demonstrated by the cancellation of two energies: the energy of the EM field and the energy of the adjacent field.

(iii) The magnitude and sign of the radiative response of a charge is affected by the composition of Poincaré cohesive forces (through the dependence on  $\vartheta$ ). The nonlinearity (which can be interpreted as charge's structure) does not affect directly the radiative response, though it affects the rest mass of the charge.

(iv) The observable radiation of a charge which may affect a test charge is described by the EM field of the point charge and can be calculated by classical formulas, it does not involve the adjacent field and does not depend on  $\vartheta_1$ .

(v) The classical field-theoretic approach allows to consistently implement mass renormalization and derive radiation response described by the generalized LAD equation, but the value of the radiative response coefficient and in particular its sign is not determined by general field-theoretic arguments and should be determined based on additional arguments or an experiment.

In this paper we consider the simplest model of a charge described by a scalar complex field  $\psi$ . A particle with spin 1/2 can be quite similarly described by a spinor field. Since our analysis does not use specific structure of the energy-momentum tensor and only uses its symmetry, our conclusions hold in this case too.

**1.1. Lagrangian formalism and field equations.** We use the following notation for the time-space 4-vector in its contravariant  $x^\mu$  and covariant  $x_\mu$  forms:

$$(8) \quad x = x^\mu = (x^0, x^1, x^2, x^3) = (ct, \mathbf{x}), \quad x_\mu = g_{\mu\nu}x^\nu = (ct, -\mathbf{x}),$$

$$(9) \quad \partial_\mu = \partial/\partial x^\mu = (c^{-1}\partial_t, \nabla), \quad \partial^\mu = \partial/\partial x_\mu = (c^{-1}\partial_t, -\nabla),$$

speed of light  $c = 1$ , and we use the common convention on the summation over the same indices. For the space 3-vector  $x^i = (x^1, x^2, x^3) = \mathbf{x}$ ,  $i = 1, 2, 3$ , we emphasize notationally by the Latin superscript its difference from 4-vector  $x^\mu$  with the Greek superscript. The metric tensor  $g_{\mu\nu}$  is defined by

$$(10) \quad \{g_{\mu\nu}\} = \{g^{\mu\nu}\} = \text{diag}[1, -1, -1, -1].$$

We consider the "charge-fields" system which describes the interaction of a single charge with two fields: its own EM field and the adjacent field generating Poincaré cohesive forces. The internal adjacent field of the charge is introduced in the model to balance the EM self-interaction of the charge and provide its stability. The charge dynamics is described by the complex scalar charge distribution  $\psi(x)$ , the EM field by its 4-potential  $A(x)$ , and the adjacent field by the 4-potential  $A^{\text{ad}}(x)$ ; the system also involves a prescribed external field with 4-potential  $A_{\text{ex}}$ . The adjacent field is solely responsible for the balancing internal interaction of the charge with itself and does not affect other charges (see Section 4 for a discussion of the interaction structure in the case of many charges).

The system "charge-fields" is described by the Lagrangian

$$(11) \quad \mathcal{L} = L_{\text{KG}} - L_{\text{em}}(A) + L_{\text{em}}(A^{\text{ad}}).$$

Here  $L_{\text{KG}}$  is the Lagrangian of a nonlinear Klein-Gordon equation:

$$(12) \quad L_{\text{KG}} = \frac{\hbar^2}{2m} \left\{ \psi_{;\mu} \psi^{;\mu} - \frac{m^2}{\hbar^2} \psi^* \psi - G(\psi^* \psi) \right\},$$

where star denotes the complex conjugation,  $m$  is a mass parameter, it is related to the mass of a charge, see [5], [6] and Section 3.1. It involves covariant derivatives

$$(13) \quad \psi^{;\mu} = \tilde{\partial}^\mu \psi = \partial^\mu \psi + \frac{iq}{\hbar} \tilde{A}^\mu \psi,$$

$$(14) \quad \tilde{\partial}^\mu = \partial^\mu + \frac{iq}{\hbar} \tilde{A}^\mu, \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_t, -\nabla), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla)$$

where  $q$  is the charge parameter and the total field potential  $\tilde{A}$  affecting the charge is defined by

$$(15) \quad \tilde{A}^\mu = A_{\text{ex}}^\mu + A^\mu - A^{\text{ad}\mu}$$

where  $A_{\text{ex}}^\mu$  is the potential of the external EM field. The Lagrangians  $L_{\text{em}}(A)$ ,  $L_{\text{em}}(A^{\text{ad}})$  for the EM field and the adjacent field are Maxwellian:

$$(16) \quad L_{\text{em}}(A) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}, \quad L_{\text{em}}(A^{\text{ad}}) = -\frac{1}{16\pi} F_{\mu\nu}^{\text{ad}} F^{\text{ad}\mu\nu}$$

with

$$(17) \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad F^{\text{ad}\mu\nu} = \partial^\mu A^{\text{ad}\nu} - \partial^\nu A^{\text{ad}\mu}.$$

Obviously, the Lagrangian  $\mathcal{L}$  is relativistic invariant. The field equations which describe dynamics of the charge-fields system are obtained as Euler-Lagrange equations for the Lagrangian  $\mathcal{L}$  and involve a nonlinear Klein-Gordon (KG) equation for the charge distribution  $\psi$  and Maxwell equations for the potentials. The KG equation has the form

$$(18) \quad \tilde{\partial}^\mu \tilde{\partial}_\mu \psi + \frac{m^2}{\hbar^2} \psi + G'(\psi^* \psi) \psi = 0$$

where  $G'(s) = dG/ds$ . The equations for the fields  $F^{\mu\nu}$ ,  $F^{\text{ad}\mu\nu}$  take the form of Maxwell equations

$$(19) \quad \partial_\mu F^{\mu\nu} = 4\pi J^\nu,$$

$$(20) \quad \partial_\mu F^{\text{ad}\mu\nu} = 4\pi J^\nu,$$

where the source current is defined by

$$(21) \quad J^\nu = -\frac{\partial L_{\text{KG}}}{\partial A_\nu} = -i\frac{q}{\hbar} \left( \frac{\partial L_{\text{KG}}}{\partial \psi_{;\nu}} \psi - \frac{\partial L_{\text{KG}}}{\partial \psi_{;\nu}^*} \psi^* \right).$$

The nonlinearity  $G$  is required to provide the localization of the charge and involves dependence on a size parameter  $a$ :  $G(b) = a^{-5} G(a^3 b)$ , the parameter  $a$  determines the scale of localization. Note that a particular form of the nonlinearity is not important in our analysis here. A typical example of the nonlinear term  $G'$  in the KG equation is given by the logarithmic expression,

$$(22) \quad G'(b) = -a^{-2} [\ln(a^3 b) + \ln \pi^{3/2} + 3], \quad b = |\psi|^2 \geq 0,$$

where  $a$  is the *charge size parameter*. For this choice of the nonlinearity the free resting charge

$$(23) \quad \psi = e^{-itm/\hbar} |\psi|$$

has a Gaussian shape, namely

$$(24) \quad |\psi| = \pi^{-3/4} a^{-3/2} e^{-|\mathbf{x}|^2 a^{-2}/2},$$

and the parameter  $a$  can be interpreted as the size of the free charge. The KG Lagrangian is gauge invariant, namely for any real  $\gamma$

$$(25) \quad L_{\text{KG}}(e^{i\gamma}\psi, e^{i\gamma}\psi_{;\mu}, e^{-i\gamma}\psi^*, e^{-i\gamma}\psi_{;\mu}^*) = L_{\text{KG}}(\psi, \psi_{;\mu}, \psi^*, \psi_{;\mu}^*).$$

Therefore the current  $J^\nu = (\rho, \mathbf{J})$  satisfies the charge conservation/continuity equation

$$(26) \quad \partial_\nu J^\nu = 0$$

which can be written in the form

$$(27) \quad \partial_t \rho + \nabla \cdot \mathbf{J} = 0.$$

The components of the 4-current defined by (21) for the KG equation have the form

$$(28) \quad \rho = -\frac{\hbar q}{m} |\psi|^2 \text{Im}(\tilde{\partial}_t \psi / \psi), \quad \mathbf{J} = \frac{\hbar q}{m} |\psi|^2 \text{Im}(\tilde{\nabla} \psi / \psi).$$

Using the fulfillment of the continuity equation we impose Lorentz gauge on the solutions of Maxwell equations. According to the continuity equation the total charge  $\int_{\mathbb{R}^3} \rho d\mathbf{x}$  is preserved. The KG equation in the absence of external fields has time-harmonic solutions of the form (23) where  $|\psi|$  satisfies the normalization condition

$$\int_{\mathbb{R}^3} |\psi|^2 d\mathbf{x} = 1.$$

An example of  $|\psi|$  which satisfies the normalization condition is given by (24). According to (28) for such harmonic solutions  $\rho = q |\psi|^2$  and the normalization condition implies that the potential of the static EM field generated by a resting charge approaches the Coulomb potential  $\frac{q}{|\mathbf{x}|}$  as  $|\mathbf{x}| \rightarrow \infty$  or  $a \rightarrow 0$ , i.e. the charge parameter  $q$  can be interpreted as the total charge of the particle.

Now we make a remark on the choice of the Lagrangian. The simplest relativistic invariant field equation which involves a mass parameter is the Klein-Gordon equation; to obtain a point-like distribution of the charge we introduce the nonlinearity. To obtain the EM field described by Maxwell's equations we introduce corresponding Lagrangian. The coupling of the charge and EM field through covariant derivatives is the minimal coupling as in [15], [1]. The simplest way to balance the EM field by another field is to introduce the adjacent field into the Lagrangian anti-symmetrically with the EM field. Therefore we think that the Lagrangian (11) describes the most natural relativistic covariant model of a distributed charge interacting with the EM field in a balanced way which provides stability of the charge and its asymptotic localization.



**1.2. Energy-momentum tensor and conservation laws .** The symmetric energy-momentum tensor (EnMT)  $\Theta^{\mu\nu}$  for the EM field is given by the formula [8], [10]:

$$(29) \quad \Theta^{\mu\nu}(F) = \frac{1}{4\pi} \left( g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

and the same formula defines  $\Theta^{\mu\nu}(F^{\text{ad}})$ . The symmetric EnMT for the KG equation has the form

$$(30) \quad \begin{aligned} T^{\mu\nu} &= \frac{\partial L_{\text{KG}}}{\partial \psi_{;\mu}} \psi^{;\nu} + \frac{\partial L_{\text{KG}}}{\partial \psi_{;\mu}^*} \psi^{;\nu*} - g^{\mu\nu} L_{\text{KG}} \\ &= \frac{\hbar^2}{2m} [\psi^{;\mu*} \psi^{;\nu} + \psi^{;\mu} \psi^{;\nu*}] - g^{\mu\nu} L_{\text{KG}}. \end{aligned}$$

The symmetric energy-momentum tensor for the charge-fields system (11) has the form

$$(31) \quad \mathcal{T}^{\mu\nu} = T^{\mu\nu} + \Theta^{\mu\nu}(F) - \Theta^{\mu\nu}(F^{\text{ad}}).$$

The crucial property of the EnMT is its symmetry

$$(32) \quad \mathcal{T}^{\mu\nu} = \mathcal{T}^{\nu\mu}.$$

The *energy and momentum conservation laws* for the system (18)-(20) have the following form:

$$(33) \quad \partial_\mu \mathcal{T}^{\mu\nu} = f^\nu.$$

Here  $f^\nu = -\partial \mathcal{L} / \partial x_\nu$  is the *Lorentz force density*, it is expressed in terms of the external EM field  $F_{\text{ex}}^{\nu\mu}$  acting on the 4-current  $J^\mu$  as follows:

$$(34) \quad f^\nu = J_\mu F_{\text{ex}}^{\nu\mu} = (f^0, \mathbf{f}).$$

Fulfillment of (31) is verified in a standard way [8], [13], [14].

The symmetric EnMT involves the energy and momentum components, namely

$$u = \mathcal{T}^{00}, \quad p^i = \mathcal{T}^{0i}, \quad i = 1, 2, 3.$$

The EnMT conservation law (33) is split into the energy conservation law

$$(35) \quad \partial_t u + \partial_j \mathcal{T}^{j0} = f^0$$

which can be rewritten using (32):

$$(36) \quad \partial_t u + \partial_j p^j = f^0,$$

with

$$(37) \quad f^0 = J_\mu F^{0\mu}(A_{\text{ex}}).$$

The momentum equation takes the form

$$(38) \quad \partial_t p^j + \partial_i \mathcal{T}^{ij} = f^j,$$

or

$$(39) \quad \partial_t \mathcal{P} + \partial_i \mathcal{T}^{ij} = \mathbf{f},$$

where the Lorentz force density is given by the formula

$$(40) \quad \mathbf{f} = f^j = J_\mu F^{j\mu}(A_{\text{ex}}) = J_0 F^{j0}(A_{\text{ex}}) + J_i F^{ji}(A_{\text{ex}}).$$

## 2. TRAJECTORY OF CONCENTRATION

We think of a point charge as a localization limit of a distributed charge. The localization means roughly speaking that charge and energy densities converge to Dirac's delta-functions. The location of this delta-function determines the trajectory  $\hat{\mathbf{r}}(t)$  of the charge. To derive the law of motion of the charge which should determine the trajectory, we assume that the energy of the charge concentrates in a small vicinity of the trajectory and the radius of the vicinity tends to zero in the localization limit. Our analysis shows that in a very general situation the fulfillment of energy-momentum conservation laws and the continuity equation uniquely determines the law of motion, namely the LAD equation; it also implies Einstein's equivalence of the concentrated energy and the mass of the charge.

Below we will show that the assumption that the energy of the charge-fields system is concentrated in a vanishing neighborhood of a trajectory implies a differential equation for the trajectory. We introduce the value  $F_\infty^{\mu\nu}(t)$  of the external field  $F^{\mu\nu}(t, \mathbf{x})$  restricted to the trajectory  $\hat{\mathbf{r}}(t)$ :

$$(41) \quad F_\infty^{\mu\nu}(t) = F_{\text{ex}}^{\mu\nu}(t, \hat{\mathbf{r}}(t)).$$

The total EnMT  $\mathcal{T}^{\mu\nu}$  is given by formula (31), which we rewrite in the form

$$(42) \quad \mathcal{T}^{\mu\nu} = T^{\mu\nu} + \tilde{\Theta}, \quad \tilde{\Theta} = \Theta^{\mu\nu}(F) - \Theta^{\mu\nu}(F^{\text{ad}}).$$

In particular, the energy density

$$(43) \quad u = \mathcal{T}^{00} = T^{00} + \Theta^{00}(F) - \Theta^{00}(F^{\text{ad}}).$$

Now we formulate our assumptions (complete mathematical details in a similar situation can be found in [6], [7]). We introduce a small ball

$$\Omega = \Omega_R(t) = \{\mathbf{x} : |\mathbf{x} - \hat{\mathbf{r}}(t)| \leq R\}$$

centered at a point of the trajectory. We assume that the energy of the charge-fields system concentrates at the trajectory  $\hat{\mathbf{r}}(t)$  on a time interval  $T_- < t < T_+$ . The field variables are split into two groups: strongly localized charge distribution  $\psi$  and EM field potentials  $A, A^{\text{ad}}$ . There is a difference in the localization of charge distributions and the EM fields. An example of a strongly localized charge is given by a Gaussian form factor

$$(44) \quad |\psi|^2 = \frac{1}{a^3 \pi^{3/2}} \exp\left(-\frac{1}{a^2} |\mathbf{x} - \hat{\mathbf{r}}|^2\right)$$

which is strongly localized in  $\Omega = \Omega_R$ ; in particular the values of the charge on  $\partial\Omega$  tend to zero very fast if  $a/R \rightarrow 0$ . The EM fields are not localized, for instance the Coulomb field potential behaves as  $\frac{q}{|\mathbf{x} - \hat{\mathbf{r}}|}$  and its values on  $\partial\Omega$  do not vanish even if  $a = 0$  when the source is completely localized as a delta-function.

We consider concentrated in  $\Omega$  energy

$$(45) \quad \bar{\mathcal{E}}(t) = \int_\Omega \mathcal{T}^{00} d\mathbf{x} = \int_\Omega u d\mathbf{x},$$

and concentrated in  $\Omega$  charge

$$(46) \quad \bar{\rho}(t) = \int_\Omega \rho d\mathbf{x}.$$

Our main assumption is that the concentrated energy and charge converge in the localization limit (5):

$$(47) \quad \bar{\mathcal{E}}(t) \rightarrow E_\infty(t) \neq 0,$$

$$(48) \quad \bar{\rho}(t) \rightarrow \bar{\rho}_\infty(t).$$

It is worth noticing that the difference  $F - F^{\text{ad}}$  does not involve a singularity, [9], therefore condition (47) mainly concerns behavior of components of the KG tensor  $T^{\mu\nu}$ .

In addition to the above condition on convergence, we assume that certain integrals asymptotically vanish. There are two types of vanishing integrals. The first type includes volume integrals over the ball  $\Omega$  or surface integrals over its boundary which involve vanishing factors. In particular, since  $(x^i - \hat{r}^i) = O(R)$ , we assume that

$$(49) \quad \int_{\Omega} (x^i - \hat{r}^i) \rho d\mathbf{x} = o(1), \quad \int_{\partial\Omega} (\mathbf{x} - \hat{\mathbf{r}}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} u d\sigma = o(1)$$

where  $\bar{\mathbf{n}}$  is the external normal to the sphere  $\partial\Omega$ ,  $\hat{\mathbf{v}} = \partial_t \hat{\mathbf{r}}$  is the trajectory-based velocity (we denote by  $O(R^p)$  such a quantity that  $O(R^p)/R^p$  remains bounded in the localization limit; notation  $o(R^p)$  means that  $o(R^p)/R^p \rightarrow 0$ ). The second type of vanishing integrals are surface integrals which are negligible because of the strong localization of  $\psi$  as in (44). Since the radius of localization of  $\psi$  is of order  $a$  and  $R \gg a$ , we assume that surface integrals over  $\partial\Omega$  that involve  $\psi$  or its derivatives as factors asymptotically vanish. For example, for  $\rho$  and  $\mathbf{J}$  given by (28) we assume that

$$(50) \quad \int_{\partial\Omega} (\hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho - \bar{\mathbf{n}} \cdot \mathbf{J}) d\sigma = o(1).$$

Similar to the above integrals that are also assumed to asymptotically vanish can be easily extracted from the calculations made in Section 3.2. Note that this kind of assumptions is natural for a point-like behavior of a charge distribution, they are obviously fulfilled for delta-functions as in (53). Detailed mathematical formulations made in similar situations and the verification of fulfillment of this kind of conditions for non-trivial examples can be found in [6], [7].

The unlocalized quantities which involve only the EM fields  $A$  or  $A^{\text{ad}}$  can make a non-vanishing contribution to the surface integrals, namely we assume that

$$(51) \quad \int_{\partial\Omega} (\hat{v}^i \bar{n}^i \mathcal{T}^{00} - \bar{n}^i \mathcal{P}^i) d\sigma = W_{\text{rad}} + o(1),$$

$$(52) \quad \int_{\partial\Omega} (\hat{v}^i \bar{n}^i \mathcal{P}^j - \bar{n}^i \mathcal{T}^{ij}) d\sigma = f_{\text{rad}}^j + o(1),$$

We make the following assumption of the localization of the charge at microscopic scales. On the sphere  $|\mathbf{x} - \hat{\mathbf{r}}(t)| = R$ , since  $\frac{a}{R} \rightarrow 0$ , the field quantities  $F^{\mu\nu}(A)$ ,  $F^{\mu\nu}(A^{\text{ad}})$  obtained as solutions of Maxwell equations (123), (124) with currents  $J^\nu = (\rho, \mathbf{J})$  can be asymptotically replaced by the solutions of Maxwell equations with the currents  $J_\infty^\nu = (\rho_\infty, \mathbf{J}_\infty)$  for the point charges:

$$(53) \quad \rho_\infty(t, \mathbf{x}) = \bar{\rho}_\infty \delta(t, \mathbf{x} - \hat{\mathbf{r}}(t)) \quad , \quad \mathbf{J}_\infty(t, \mathbf{x}) = \hat{\mathbf{v}} \bar{\rho}_\infty \delta(t, \mathbf{x} - \hat{\mathbf{r}}(t)).$$

Namely, we assume convergence on  $\partial\Omega = \{|\mathbf{x} - \hat{\mathbf{r}}(t)| = R\}$  in the localization limit (5) of the advanced and retarded potentials:

$$(54) \quad (\mathcal{G}_{\text{ret}} J^\nu)_{\mathbf{x} \in \partial\Omega} \rightarrow (\mathcal{G}_{\text{ret}} J_\infty^\nu)_{\mathbf{x} \in \partial\Omega}, \quad (\mathcal{G}_{\text{adv}} J^\nu)_{\mathbf{x} \in \partial\Omega} \rightarrow (\mathcal{G}_{\text{adv}} J_\infty^\nu)_{\mathbf{x} \in \partial\Omega}.$$

We will use the above convergence to evaluate radiative terms  $W_{\text{rad}}$  and  $f_{\text{rad}}^j$  in (51), (52).

### 3. DERIVATION OF NEWTON'S LAW FOR THE TRAJECTORY

In this section we derive the relativistic version of Newton's law for the trajectory  $\hat{\mathbf{r}}(t)$ . We provide principal steps, complete mathematical details presented in a similar situation can be found in [6], [7]. The equations are determined uniquely by the assumptions made in the previous section.

We start with assigning a position to the distributed charge by using its energy density  $u = \mathcal{T}^{00}$ . We define the *energy center*  $\mathbf{r}(t)$  by the formula

$$(55) \quad \mathbf{r}(t) = \bar{\mathcal{E}}(t)^{-1} \int_{\Omega} \mathbf{x} u(t, \mathbf{x}) \, d\mathbf{x}$$

where concentrated energy  $\bar{\mathcal{E}}(t)$  is defined by (45). One can easily prove using (47) that

$$(56) \quad \mathbf{r}(t) \rightarrow \hat{\mathbf{r}}(t).$$

From the conservation laws for the energy-momentum and the continuity equation we derive in subsection 3.2 the following differential equations for the trajectory and the energy  $\bar{\mathcal{E}}_\infty$  concentrated at it:

$$(57) \quad \partial_t \bar{\mathcal{E}}_\infty(t) = \hat{v}^j f_\infty^j + W_{\text{rad}},$$

$$(58) \quad \partial_t (\bar{\mathcal{E}}_\infty(t) \hat{v}^j) = f_\infty^j + f_{\text{rad}}^j.$$

Here  $\hat{v}^j = \partial_t \hat{\mathbf{r}}^j(t)$ , the Lorentz force  $f_\infty^j$  is generated by the external field according to formula (77) and  $f_{\text{rad}}^j$  is radiation force defined by (52). Obviously (58) has the form of Newton's law with the mass  $M = \bar{\mathcal{E}}_\infty(t)$  which implies Einstein's formula on equivalence of mass and energy. Note that we derive this formula in a regime with acceleration where Newtonian definition of mass is applicable. A similar derivation of Einstein's formula in the regime without radiation is given in [5], [6]. Examples of accelerating localized solutions of KG equations are also given there. From equations (57) and (58) we will derive the LAD equation in subsection 3.4.2.

**3.1. Rest mass of a charge and mass renormalization.** In this section we define the rest mass as an integral of motion of the system (57), (58). We also demonstrate the mass renormalization as cancellation of two energies: the energy of EM field and the energy of the adjacent field.

We multiply equations (58) and (57) by  $\bar{\mathcal{E}}_\infty \hat{v}^j$  and  $\bar{\mathcal{E}}_\infty$  respectively and after subtraction obtain

$$(59) \quad \partial_t (\bar{\mathcal{E}}_\infty(t) \hat{v}^j) \bar{\mathcal{E}}_\infty \hat{v}^j - \bar{\mathcal{E}}_\infty \partial_t \bar{\mathcal{E}}_\infty(t) = \bar{\mathcal{E}}_\infty (\hat{v}^j f_{\text{rad}}^j - W_{\text{rad}}).$$

We assume that the following "radiative balance condition" for the radiative terms is fulfilled:

$$(60) \quad \hat{v}^j f_{\text{rad}}^j - W_{\text{rad}} = 0.$$

Later in this section we obtain formula (114) which shows that the radiative balance condition (60) is fulfilled under the assumptions (54). From (59) and (60) we obtain the following rest mass equation:

$$(61) \quad \partial_t \left[ \bar{\mathcal{E}}_\infty^2(t) \left( 1 - |\hat{\mathbf{v}}|^2 \right) \right] = 0.$$

The expression in brackets equals a constant which we denote  $M_0^2$ , where the constant  $M_0$  is the rest mass. As a result we obtain a familiar explicit expression for the time dependence of concentrated energy:

$$(62) \quad \bar{\mathcal{E}}_\infty(t) = \left( 1 - |\hat{\mathbf{v}}|^2 \right)^{-1/2} M_0.$$

Equation (58) takes the form of relativistic version of Newton's law

$$(63) \quad \partial_t \left( M_0 \left( 1 - |\hat{\mathbf{v}}|^2 \right)^{-1/2} \hat{\mathbf{v}}^j \right) = f_\infty^j + f_{\text{rad}}^j$$

where  $M_0$  is the rest mass of the charge. *The rest mass  $M_0$  is the observable mass of the charge and its existence signifies the mass renormalization.* Obviously, the rest mass is not prescribed but emerges as an integral of motion of a system (57), (58).

According to (43) the concentrated energy is given by the formula

$$(64) \quad \bar{\mathcal{E}}_\infty = \lim_{R \rightarrow 0, \frac{a}{R} \rightarrow 0} \int_\Omega \left[ T^{00} + \Theta^{00}(A) - \Theta^{00}(A^{\text{ad}}) \right] d\mathbf{x}.$$

The crucial property of the radiation field  $F_{\text{rad}} = F_{\text{ret}} - F_{\text{adv}}$  proved by Dirac [9] is its regularity. Thanks to the regularity the tensor  $\tilde{\Theta}^{00} = \Theta^{00}(A) - \Theta^{00}(A^{\text{ad}})$  which can be represented by (94) has a mild singularity at the trajectory which not stronger than  $|\mathbf{x} - \hat{\mathbf{r}}|^{-2}$  at the center of  $\Omega$ . Therefore it does not contribute to the limit in (64) and

$$(65) \quad \bar{\mathcal{E}}_\infty = \lim_{R \rightarrow 0, \frac{a}{R} \rightarrow 0} \int_\Omega T^{00} d\mathbf{x}.$$

At the same time both the EM energy and the energy of the adjacent field which enter (64) turn into infinity in the localization limit:

$$\int_\Omega \Theta^{00}(A) d\mathbf{x} \rightarrow \infty, \quad \int_\Omega \Theta^{00}(A^{\text{ad}}) d\mathbf{x} \rightarrow \infty$$

as  $R \rightarrow 0, \frac{a}{R} \rightarrow 0$ . Therefore, formulas (64) and (65) provide an explicit cancellation of the infinite energies which constitutes the mass renormalization. In a contrast to the customary treatment of the mass renormalization, [9], [14], [16], [19], [21], [22], where such a cancellation is postulated, we demonstrate the cancellation explicitly.

Now we find the value of the rest mass  $M_0$  in a regime which originates from the uniform motion. If the motion is uniform for  $t < 0$  with the initial velocity  $\hat{\mathbf{v}}_{\text{init}}$  and the motion is accelerated only for  $t > 0$ , the value of the rest mass can be determined based on the analysis of uniform motion of a charge which is described explicitly in [4], [5], [6]. Calculations made there show that

$$(66) \quad \bar{\mathcal{E}}_\infty = M_0 \left( 1 - |\hat{\mathbf{v}}_{\text{init}}|^2 \right)^{-1/2}, \quad M_0 = mc^2 \left( 1 + \Theta_0 \frac{a_C^2}{a^2} \right)$$

where  $a_C = \frac{\hbar}{mc} = \frac{\hbar}{m}$  is the reduced Compton wavelength and the factor  $\Theta_0$  depends on the nonlinearity. For example,  $\Theta_0 = 1/2$  for the logarithmic nonlinearity (22) corresponding to the Gaussian ground state (44). Note that if  $\frac{ac}{a}$  converges to a non-zero value, the nonlinearity provides a non-vanishing contribution to the rest mass, and if  $\frac{ac}{a} \rightarrow 0$  the rest mass  $M_0 = m$  coincides with the mass parameter in the KG equation. For a discussion of the concept of mass see [5]. The total charge  $\bar{\rho}_\infty$  defined by (46), (28) and (68) is given by the formula, [5], [6]:

$$\bar{\rho}_\infty = q$$

where  $q$  is the charge parameter in the KG equation defined by (18), (13). For the electron we set  $q = e$ .

**3.2. Convergence in localization limit.** Now we proceed to the derivation of equations (57) and (58).

**3.2.1. Charge convergence.** First we show that defined by (46) concentrated charge  $\bar{\rho}_n$  converges to a constant. We integrate the continuity equation:

$$(67) \quad \bar{\rho}_n(t) - \bar{\rho}_n(t_0) = \int_{t_0}^t \int_{\partial\Omega} (\hat{\mathbf{v}} \cdot \bar{\mathbf{n}}\rho - \bar{\mathbf{n}} \cdot \mathbf{J}) \, d\sigma dt'.$$

Since the integrals over boundary converge to zero, we obtain that

$$(68) \quad \bar{\rho}_n(t) \rightarrow \bar{\rho}_\infty \quad \text{for } T_- < t < T_+$$

where  $\bar{\rho}_\infty$  does not depend on  $t$ . The above relation can be written in the form

$$(69) \quad \bar{J}^0 = \bar{\rho}_\infty \hat{v}^0 + o(1)$$

where  $\hat{v}^0 = 1$ .

**3.2.2. Lorentz force convergence.** Now we show that averaged Lorentz force density converges to the Lorentz force acting on a point charge. Since  $F^{\mu\nu} - F_\infty^{\mu\nu} = o(1)$  in  $\Omega$ , we conclude that

$$(70) \quad \begin{aligned} \int_{\Omega} f^\mu \, d\mathbf{x} &= \int_{\Omega} F^{\mu\nu} J_\nu \, d\mathbf{x} \\ &= \int_{\Omega} F_\infty^{\mu\nu}(t) J_\nu \, d\mathbf{x} + \int_{\Omega} (F^{\mu\nu} - F_\infty^{\mu\nu}) J_\nu \, d\mathbf{x} = F_\infty^{\mu\nu}(t) \bar{J}_\nu + o(1). \end{aligned}$$

Here

$$(71) \quad \bar{J}^\mu = \int_{\Omega} J^\mu \, d\mathbf{x};$$

in the particular case  $\mu = 0$  the above formula turns into (46). To find an expression for  $\bar{J}^j$   $j = 1, 2, 3$ , we multiply the continuity equation (27) by the vector  $\mathbf{x} - \hat{\mathbf{r}}$  and using the commutation relation  $\partial_j(x^i \varphi) - x^i \partial_j \varphi = \delta_{ij} \varphi$  we obtain the following expression for  $J^j$ :

$$(72) \quad \partial_t(\rho(\mathbf{x} - \hat{\mathbf{r}})) + \partial_t \hat{\mathbf{r}} \rho + \partial_i \left( (\mathbf{x} - \hat{\mathbf{r}})^j J^i \right) = J^j.$$

Integrating we arrive to the equation

$$(73) \quad \bar{J}^j = \hat{v}^j \bar{\rho} + \partial_t \int_{\Omega} (x^i - \hat{r}^i) \rho \, d\mathbf{x} + \int_{\partial\Omega} (x^i - \hat{r}^i) (\bar{\mathbf{n}} \cdot \mathbf{J} - \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} \rho) \, d\sigma,$$

where  $(x^i - \hat{r}^i) = o(1)$  in  $\Omega$ . Together with (69) the above equation implies that

$$(74) \quad \bar{J}^\nu = \bar{\rho}_\infty \hat{v}^\nu + \partial_t o(1) + o(1)$$

Therefore (70) implies an expression for the concentrated Lorentz force density:

$$(75) \quad \int_{\Omega} f^\mu d\mathbf{x} = \bar{\rho}_\infty F_\infty^{\mu\nu} \hat{v}_\nu + \partial_t o(1) + o(1).$$

We can write it in the form

$$(76) \quad \int_{\Omega} f^\mu d\mathbf{x} = f_\infty^\mu + o(1) + \partial_t o(1)$$

where the components of the Lorentz force  $f_\infty^j$  are given by

$$(77) \quad f_\infty^j = \bar{\rho}_\infty F_\infty^{j0}(t) + \bar{\rho}_\infty F_\infty^{ji}(t) \hat{v}_i$$

and

$$(78) \quad f_\infty^0 = \bar{\rho}_\infty F_\infty^{0\nu} \hat{v}_\nu$$

Note that since  $F_\infty^{ji}$  is skew-symmetric,

$$f_\infty^j \hat{v}^j = \bar{\rho}_\infty F_\infty^{j0}(t) \hat{v}^j + \bar{\rho}_\infty F_\infty^{ji}(t) \hat{v}_i \hat{v}^j = \bar{\rho}_\infty F_\infty^{j0}(t) \hat{v}^j = -\bar{\rho}_\infty F_\infty^{0\nu} \hat{v}^\nu$$

and we can rewrite (78) in the form

$$(79) \quad f_\infty^0 = \bar{\rho}_\infty F_\infty^{0\nu} \hat{v}_\nu = -\bar{\rho}_\infty F_\infty^{0\nu} \hat{v}^\nu = f_\infty^j \hat{v}^j.$$

**3.2.3. Momentum convergence.** Multiplying energy equation (36) by  $(\mathbf{x} - \mathbf{r})$ , we obtain

$$(80) \quad \partial_t ((\mathbf{x} - \mathbf{r}) u) + u \partial_t \mathbf{r} = -(\mathbf{x} - \mathbf{r}) \partial_j p^j + (\mathbf{x} - \mathbf{r}) f^0$$

where  $(\mathbf{x} - \mathbf{r}) \partial_j p^j = \partial_j ((\mathbf{x} - \mathbf{r}) p^j) - p^j$ . We obtain from the relation (80) an expression for  $\mathcal{P} = p^j$  and its integral:

$$(81) \quad \begin{aligned} \int_{\Omega} \mathcal{P} d\mathbf{x} &= \int_{\Omega} \partial_t ((\mathbf{x} - \mathbf{r}) u) d\mathbf{x} + \partial_t \mathbf{r} \int_{\Omega} u d\mathbf{x} \\ &\quad + \int_{\partial\Omega} (\mathbf{x} - \mathbf{r}) n^i p^i d\sigma - \int_{\Omega} (\mathbf{x} - \mathbf{r}) f^0 d\mathbf{x}. \end{aligned}$$

From the definition of the energy center  $\mathbf{r}$  we infer that

$$(82) \quad \int_{\Omega} \partial_t ((\mathbf{x} - \mathbf{r}) u) d\mathbf{x} + \int_{\partial\Omega} (\mathbf{x} - \mathbf{r}) \hat{\mathbf{v}} \cdot \bar{\mathbf{n}} u d\sigma = 0,$$

therefore

$$\int_{\Omega} \partial_t ((\mathbf{x} - \mathbf{r}) u) d\mathbf{x} = o(1).$$

Hence, taking into account that  $(\mathbf{x} - \mathbf{r}) = O(R)$  in  $\Omega$ , we conclude that

$$(83) \quad \int_{\Omega_n} \mathcal{P} d\mathbf{x} = \partial_t \mathbf{r} \bar{\mathcal{E}} + o(1).$$

3.2.4. *Energy balance.* Integrating energy equation (36) with respect to  $x$  and  $t$  we obtain the equation

$$(84) \quad \bar{\mathcal{E}}(t) - \bar{\mathcal{E}}(t_0) = \int_{t_0}^t \int_{\partial\Omega} (\hat{\mathbf{v}} \cdot \bar{\mathbf{n}}u - \bar{\mathbf{n}} \cdot \mathcal{P}) d\sigma dt' + \int_{t_0}^t \int_{\Omega} f^0 d\mathbf{x} dt'$$

where according to (76) with  $\mu = 0$

$$\int_{t_0}^t \int_{\Omega} f^0 d\mathbf{x} dt' = f_{\infty}^{\mu} + o(1).$$

Applying (51) we obtain

$$(85) \quad \bar{\mathcal{E}}(t) - \bar{\mathcal{E}}(t_0) = \int_{t_0}^t f_{\infty}^0 dt' + \int_{t_0}^t W_{\text{rad}} dt' + o(1).$$

3.2.5. *Momentum balance.* Integrating the momentum conservation equation (39) over  $\Omega$  ( $\hat{\mathbf{r}}(t), R_n$ ) =  $\Omega$  and with respect to time we obtain

$$(86) \quad \int_{\Omega} p^j(t, \mathbf{x}) d\mathbf{x} - \int_{\Omega} p^j(t_0, \mathbf{x}) d\mathbf{x} - \int_{t_0}^t \int_{\Omega} f^j d\mathbf{x} dt' + \int_{t_0}^t \int_{\partial\Omega_n} (\bar{n}^i T^{ij} - \hat{v}^i \bar{n}^i p^j) d\sigma dt' = 0$$

where according to (76)

$$\int_{t_0}^t \int_{\Omega} f^j d\mathbf{x} dt' = f_{\infty}^j + o(1).$$

Applying (83) and (52) we obtain

$$(87) \quad \bar{\mathcal{E}}(t) \partial_t r^j(t) - \bar{\mathcal{E}}_n(t_0) \partial_t r^j(t_0) = \int_{t_0}^t f_{\infty}^j dt' dt'' + f_{\text{rad}}^j + o(1)$$

with the Lorentz force given by (77).

3.2.6. *The necessary conditions on energy concentration.* Passing to the localization limit in (85) we obtain

$$\bar{\mathcal{E}}_{\infty}(t) - \bar{\mathcal{E}}_{\infty}(t_0) = \int_{t_0}^t f_{\infty}^0 dt' + \int_{t_0}^t W_{\text{rad}} dt'.$$

Using (79) we obtain equation (57) for the limit energy. Now we derive Newton's law (58). From (87), after obvious manipulations, we obtain

$$r^j(t) - r^j(t_0) - \int_{t_0}^t \frac{\bar{\mathcal{E}}(t_0)}{\bar{\mathcal{E}}(t)} \partial_t r^j(t_0) dt'' = \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}(t'')} \int_{t_0}^{t''} f_{\infty}^j dt' dt'' + o(1)$$

and using (56) and (47) we obtain passing to the localization limit:

$$\hat{r}^j(t) - \hat{r}^j(t_0) - \partial_t \hat{r}^j(t_0) \int_{t_0}^t \frac{\bar{\mathcal{E}}_{\infty}(t_0)}{\bar{\mathcal{E}}_{\infty}(t)} dt'' = \int_{t_0}^t \frac{1}{\bar{\mathcal{E}}_{\infty}(t'')} \int_{t_0}^{t''} f_{\infty}^j dt' dt''.$$

From this integral equation we obtain differential equation (58).



**3.3. Radiation force and power for a localized charge.** To derive LAD equation we have to calculate the radiation power  $W_{\text{rad}}$  and radiation force  $f_{\text{rad}}^j$  in equations (60) and (63). Note that since  $\psi$  is localized, the KG tensor  $T^{\mu\nu}$  given by (30) asymptotically vanish and

$$\int_{\partial\Omega} (\hat{v}^i \bar{n}^i T^{00} - \bar{n}^i T^{0i}) d\sigma = o(1).$$

Therefore definition (51), (52) can be rewritten in the form

$$(88) \quad W_{\text{rad}} = \lim_{R \rightarrow 0} \int_{\partial\Omega} \left( \hat{v}^i \bar{n}^i \tilde{\Theta}^{00} - \bar{n}^i \tilde{\Theta}^{0i} \right) d\sigma,$$

$$(89) \quad f_{\text{rad}}^j = - \lim_{R \rightarrow 0} \int_{\partial\Omega} \left( \bar{n}^i \tilde{\Theta}^{ij} - \hat{v}^i \bar{n}^i \tilde{\Theta}^{0j} \right) d\sigma.$$

The stress tensor  $\tilde{\Theta}^{\mu\nu}$  is defined by formula (42) with the tensor  $\Theta^{\mu\nu}(F)$  defined by (29) in terms of solutions of Maxwell equations (19), (20). We assume that the solution of the Maxwell equations is given by the formula (3), (4) where  $\mathcal{G}_{\text{ret}}$  and  $\mathcal{G}_{\text{adv}}$  the solution operators defined as integral operators with retarded and advanced Green functions respectively, in particular

$$(90) \quad \mathcal{G}_{\text{ret}} J^\nu(t, \mathbf{x}) = \int \frac{[J^\nu(t', \mathbf{x}')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}',$$

where the symbol  $[\cdot]_{\text{ret}}$  means that the quantity in the square brackets is to be evaluated at the retarded time

$$(91) \quad t' = t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|.$$

It is important to notice that the sources in equations (19) and (20) are the same, and only the coefficients  $\vartheta_0, \vartheta_1$  which determine the composition of Green functions are different. Now we find the integrals (88), (89) under the assumption (54). Notice that the difference  $F_{\text{ret}} - F_{\text{adv}}$  for the point sources does not have a singularity at  $\hat{\mathbf{r}}(t)$ , [9], and is a continuous function.

The tensor  $\Theta^{\mu\nu}(F)$  defined by (29) is quadratic with respect to field components  $F^{\alpha\beta}$ :  $\Theta^{\mu\nu}(F) = \Theta^{\mu\nu}(FF)$ . Taking into account (3), (4) we see that

$$(92) \quad \tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} \left( (F_{\text{adv}} + \vartheta_0 F_{\text{rad}})^2 \right) - \Theta^{\mu\nu} \left( (F_{\text{adv}} + \vartheta_1 F_{\text{rad}})^2 \right)$$

where

$$(93) \quad F_{\text{rad}} = F_{\text{ret}} - F_{\text{adv}}$$

Expanding the quadratic tensors  $\Theta^{\mu\nu}$  we obtain

$$(94) \quad \tilde{\Theta}^{\mu\nu} = (\vartheta_0 - \vartheta_1) \tilde{\Theta}_1^{\mu\nu} + (\vartheta_0^2 - \vartheta_1^2) \Theta^{\mu\nu} \left( (F_{\text{rad}})^2 \right)$$

where

$$(95) \quad \tilde{\Theta}_1^{\mu\nu} = \Theta^{\mu\nu}(F_{\text{adv}}, F_{\text{rad}}) + \Theta^{\mu\nu}(F_{\text{rad}}, F_{\text{adv}})$$

Since  $F_{\text{rad}}$  has no singularity at  $\hat{\mathbf{r}}(t)$ , the integral of  $\Theta^{\mu\nu} \left( (F_{\text{rad}})^2 \right)$  gives a vanishing contribution as  $R \rightarrow 0$  and we can replace  $\tilde{\Theta}^{\mu\nu}$  by  $\tilde{\Theta}_1^{\mu\nu}$  in (88), (89). Note that formula (95) already indicates that limits (88), (89) are well-defined since the only singularity comes from  $F_{\text{adv}}$  and it has leading order  $R^{-2}$  whereas the sphere area is  $4\pi R^2$ . But still we need to evaluate the integrals involving  $\tilde{\Theta}_1^{\mu\nu}$ .

To evaluate  $\tilde{\Theta}_1^{\mu\nu}$  we use expressions obtained by Dirac [9] for the fields  $F_{\text{rad}} = F_{\text{ret}} - F_{\text{adv}}$ ,  $F_{\text{ret}}$  and  $F_{\text{adv}}$ . The expressions are given in the following section.

**3.4. Energy momentum tensors of EM field near trajectory.** Here we use expressions obtain by Dirac [9] for the EnMT of the field generated by a point charge in our notation. We have to note that Dirac used notation  $A_\nu$  for the EM potential whereas we use notation  $A^\nu$  for the same potential and tensor  $F^{\mu\nu}$  in Dirac's notation equals  $F_{\mu\nu}$  in our notation, hence we rewrite the formulas in our notation. We also denote by  $(vw)$  scalar product of 4-vectors given by (2),  $(vw) = v_0 w_0 - \mathbf{v} \cdot \mathbf{w}$  where  $\mathbf{v} \cdot \mathbf{w}$  is the usual dot product in 3D space. We assume a parametrization of the trajectory

$$(t, \hat{\mathbf{r}}(t)) = (r^0, \hat{\mathbf{r}}(t)) = z(s) = (z_0(s), \mathbf{z}(s))$$

by the proper time  $s$  so that

$$v^\alpha = \partial_s r^\alpha = \dot{z}^\alpha(s) = (v^0, \mathbf{v})$$

satisfies the normalization

$$(vv) = v_0^2 - |\mathbf{v}|^2 = 1.$$

The normalization implies identities

$$(v\dot{v}) = 0, \quad (v\ddot{v}) + (\dot{v}\dot{v}) = 0$$

where  $\dot{v} = \partial_s v$ . Note that

$$\hat{v}_i = v_i/v_0.$$

We fix a point of a trajectory and a sphere  $\partial\Omega$  of radius  $R$  centered at it. We need to find values of the tensors  $\tilde{\Theta}_1^{\mu\nu}$  on the sphere  $\partial\Omega$ . We choose the the coordinates so that the center of the sphere has zero coordinates. We also for the calculation take the value of  $s$  so that  $s = 0$  corresponds to the center of the sphere. Therefore a point  $x = (t, \mathbf{x})$  on the sphere  $\partial\Omega$  satisfies the equation equation  $t = 0, |\mathbf{x}| = R$  and the external normal to the sphere can be written as  $\bar{\mathbf{n}} = \frac{\mathbf{x}}{R}$ . Taking into account (94) we can write (88), (89) in the form

$$(96) \quad W_{\text{rad}} = \vartheta \lim_{R \rightarrow 0} \frac{1}{R} \int_{\partial\Omega} \left( \frac{v^i}{v_0} x^i \tilde{\Theta}_{\parallel}^{00} - x^i \tilde{\Theta}_{\parallel}^{0i} \right) d\sigma,$$

$$(97) \quad f_{\text{rad}}^j = -\vartheta \lim_{R \rightarrow 0} \frac{1}{R} \int_{\partial\Omega} \left( x^i \tilde{\Theta}_{\parallel}^{ij} - \frac{v^i}{v_0} x^i \tilde{\Theta}_{\parallel}^{0j} \right) d\sigma,$$

where  $\vartheta = \vartheta_0 - \vartheta_1$ , and  $\tilde{\Theta}_{\parallel}^{\mu\nu}$  is the leading part of the tensor (95):

$$\tilde{\Theta}_1^{\mu\nu} = \tilde{\Theta}_{\parallel}^{\mu\nu} + o(R^{-2}).$$

The tensor  $\tilde{\Theta}_{\parallel}^{\mu\nu}$  is explicitly written below in (103) and now we describe its derivation. According to (95)  $\tilde{\Theta}_1^{\mu\nu}$  involves products of components of  $F_{\text{rad}}$  and  $F_{\text{adv}}$ . Now we write the leading terms of the expressions for  $F_{\text{rad}}$  and  $F_{\text{adv}}$  from [9] which contribute to the integral (88), (89). The field  $F_{\text{rad}\mu\nu}$  has no singularity hence the leading terms for  $F_{\text{rad}\mu\nu}$  are of zero order in  $R$ , and the leading terms for  $F_{\text{adv}\mu\nu}$  are of order  $R^{-2}$ . The leading term of formula (12) in [9] takes the form

$$(98) \quad F_{\text{rad}\mu\nu} = \frac{4e}{3} (\ddot{v}^\mu v^\nu - \ddot{v}^\nu v^\mu).$$

where  $e$  is the value of a point charge. The leading part  $F_{0\mu\nu}$  of the tensor  $F_{\text{adv}\mu\nu}$  (which coincides with the leading part of  $F_{\text{ret}\mu\nu}$ ) is given by the expression

$$(99) \quad F_{0\mu\nu} = -e\epsilon^{-3} (v^\mu \gamma^\nu - v^\nu \gamma^\mu)$$

obtained from formula (60) in [9]. Now we explain the notation in this formula. The 4-vector  $\gamma = \gamma^\nu$  is defined by the relation

$$x^\mu = z^\mu(s_0) + \gamma^\mu$$

where  $s_0$  is such that

$$(\gamma \dot{z}(s_0)) = 0.$$

The 4-vector  $\gamma$  is space-like:

$$-\epsilon^2 = (\gamma\gamma) = \gamma_0^2 - \gamma^2$$

where  $\epsilon > 0$  is the same as in (99).

Obviously  $\gamma = O(R)$  on  $\partial\Omega$  and to find the leading term it is sufficient to find  $s_0$  and  $\gamma$  with accuracy  $o(R)$ . The trajectory can be approximated by the tangent straight line with the accuracy  $O(R^2)$ , and we can determine  $s_0$  using this approximation. An elementary calculation produces

$$(100) \quad s_0 = x_0 \dot{z}_0 - \mathbf{x} \cdot \mathbf{v}$$

where  $\mathbf{v} = \dot{\mathbf{z}}$  and

$$(101) \quad \gamma = x - \dot{z}(x_0 \dot{z}_0 - \mathbf{x} \cdot \mathbf{v}) = x - (x\dot{z})\dot{z},$$

$x_0 = 0$ . Calculating  $(\gamma\gamma)$  we obtain

$$(102) \quad \epsilon = \left( R^2 + (\mathbf{x} \cdot \mathbf{v})^2 \right)^{1/2}.$$

Therefore all the terms in (99) are explicitly expressed in terms of  $\mathbf{x}$  and the derivatives of trajectory coordinates. The derivatives have to be calculated at  $s = s_0$ , but for the leading term we can take their values at  $s = 0$  that is at the center of the sphere  $\partial\Omega$ .

We have to evaluate  $\tilde{\Theta}_1^{\mu\nu}$  defined by (95) where  $\Theta^{\mu\nu}(F)$  is defined by (29). A straightforward computation produces

$$(103) \quad \tilde{\Theta}_1^{\mu\nu} = \frac{1}{4\pi} g^{\mu\alpha} F_{0\alpha\beta} F_{\text{rad}}^{\beta\nu} + \frac{1}{4\pi} g^{\mu\alpha} F_{\text{rad}\alpha\beta} F_0^{\beta\nu} + \frac{1}{8\pi} g^{\mu\nu} F_{0\alpha\beta} F_{\text{rad}}^{\alpha\beta}.$$

Now we substitute expressions (98) and (99) into the above formula. After straightforward calculations we obtain the following expression for the leading term of  $\tilde{\Theta}_1^{\mu\nu}$  on  $\partial\Omega$ :

$$(104) \quad -\frac{3\pi}{e^2 \epsilon^{-3}} \tilde{\Theta}_{\text{ll}}^{\mu\nu} = [2v_\mu v_\nu \ddot{v}_\beta - v_\beta (v_\nu \ddot{v}_\mu + v_\mu \ddot{v}_\nu) + g^{\mu\nu} ((v\ddot{v}) v_\beta - \ddot{v}_\beta)] x^\beta \\ + x_\mu (\ddot{v}_\nu - (v\ddot{v}) v_\nu) + x_\nu (\ddot{v}_\mu - (v\ddot{v}) v_\mu)$$

with  $x_0 = 0$  on  $\partial\Omega$ .

3.4.1. *Evaluation of the integrals.* According to (104) the integrals (96), (97) over the sphere  $\partial\Omega$  involve integrals of the form

$$(105) \quad \begin{aligned} \Xi(x^i, x^j) &= \Xi(x_i, x_j) = \frac{1}{R} \int_{\partial\Omega} x_i x_j \epsilon^{-3} d\sigma \\ &= \frac{1}{R} \int_{|\mathbf{x}|=R} x_i x_j \left( R^2 + (\mathbf{x} \cdot \mathbf{v})^2 \right)^{-3/2} d\sigma. \end{aligned}$$

Evaluating the above integral (see Appendix) we obtain that

$$(106) \quad \Xi(x_i, x_j) = \frac{v_i v_j}{\mathbf{v}^2} [\Xi_{\parallel} - \Xi_{\perp}] + \delta_{ij} \Xi_{\perp}.$$

The coefficients  $\Xi_{\parallel}, \Xi_{\perp}$  which depend only on  $|\mathbf{v}|$  are calculated in the Appendix, where the following useful formula is obtained:

$$(107) \quad \Xi_{\parallel} + 2\Xi_{\perp} = \frac{4\pi}{\sqrt{|\mathbf{v}|^2 + 1}} = \frac{4\pi}{v_0}.$$

When we use formula (106) it is convenient to evaluate separately terms at coefficients  $[\Xi_{\parallel} - \Xi_{\perp}]$  and  $\Xi_{\perp}$ . After elementary but tedious calculations we obtain expressions for integrals in (96), (97):

$$(108) \quad \begin{aligned} -\frac{3\pi}{e^2} \frac{1}{R} \int x^i \tilde{\Theta}_{\parallel}^{ij} \epsilon^{-3} d\sigma &= [\Xi_{\parallel} - \Xi_{\perp}] [v^j \ddot{\mathbf{v}} \cdot \mathbf{v} - \mathbf{v}^2 \ddot{v}^j - (\ddot{v}^j - v^j (v\ddot{v}))] \\ &\quad + \Xi_{\perp} [\ddot{\mathbf{v}} \cdot \mathbf{v} v^j - \mathbf{v}^2 \ddot{v}^j - 3\ddot{v}^j + 3v^j (v\ddot{v})], \end{aligned}$$

$$(109) \quad -\frac{3\pi}{e^2} \frac{1}{R} \int \hat{v}^i x^i \tilde{\Theta}_{\parallel}^{0j} \epsilon^{-3} d\sigma = \Xi_{\parallel} [-\ddot{\mathbf{v}} \cdot \mathbf{v} v^j + \mathbf{v}^2 \ddot{v}^j],$$

$$(110) \quad -\frac{3\pi}{e^2} \frac{1}{R} \int \tilde{\Theta}_{\parallel}^{0j} x^j \epsilon^{-3} d\sigma = -2\Xi_{\perp} (\mathbf{v}^2 \ddot{v}_0 - v_0 \ddot{\mathbf{v}} \cdot \mathbf{v}),$$

$$(111) \quad -\frac{3\pi}{e^2} R^{-1} \frac{1}{v_0} \int (v^i x^i \Theta_{\parallel}^{00}) \epsilon^{-3} d\sigma = \Xi_{\parallel} (-v_0 \ddot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v}^2 \ddot{v}_0).$$

Substituting the above expressions we obtain

$$(112) \quad W_{\text{rad}} = -\frac{e^2}{3\pi} \vartheta (\Xi_{\parallel} + 2\Xi_{\perp}) (-v_0 \ddot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v}^2 \ddot{v}_0),$$

$$(113) \quad f_{\text{rad}}^j = \frac{e^2}{3\pi} \vartheta (\Xi_{\parallel} + 2\Xi_{\perp}) [\ddot{v}^j - v^j (v\ddot{v})].$$

Now we are able to check the radiative balance condition (60):

$$(114) \quad \begin{aligned} v_0 W_{\text{rad}} - \mathbf{v} \cdot \mathbf{f}_{\text{rad}} &= -\frac{e^2}{3\pi} \vartheta (\Xi_{\parallel} + 2\Xi_{\perp}) [-v_0^2 \ddot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v}^2 \ddot{v}_0 v_0 + \ddot{\mathbf{v}} \cdot \mathbf{v} - \mathbf{v}^2 (v\ddot{v})] \\ &= -\frac{e^2}{3\pi} \vartheta (\Xi_{\parallel} + 2\Xi_{\perp}) [-v_0^2 \ddot{\mathbf{v}} \cdot \mathbf{v} + \ddot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v}^2 \ddot{\mathbf{v}} \cdot \mathbf{v}] = 0. \end{aligned}$$

Using (107) we rewrite the radiative force as follows:

$$(115) \quad f_{\text{rad}}^j = \frac{4e^2}{3v_0} \vartheta [\ddot{v}^j + v^j (v\ddot{v})].$$

3.4.2. *Generalized LAD equation.* Rewriting (63) in terms of proper time  $s$  we obtain

$$(116) \quad \partial_s v^j = v_0 f_\infty^j + \frac{4e^2}{3} \vartheta [\ddot{v}^j + v^j (\dot{v}\dot{v})].$$

Equation (57), where according to (79)  $\hat{v}^j f_\infty^j = \bar{\rho}_\infty F_\infty^{0\nu} \hat{v}_\nu = v_\mu \bar{\rho}_\infty F_\infty^{0\mu} / v^0$ , takes the form

$$(117) \quad \begin{aligned} \partial_s (v_0 M_0) &= v_0 v^j f_\infty^j + v_0 W_{\text{rad}} = v_\mu \bar{\rho}_\infty F_\infty^{0\mu} \\ -\frac{4e^2 \vartheta}{3v_0} (-v_0^2 \ddot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v}^2 \ddot{v}_0 v_0) &= v_\mu \bar{\rho}_\infty F_\infty^{0\mu} + \frac{4e^2}{3} \vartheta (\ddot{v}_0 - v_0 (\ddot{v}v)). \end{aligned}$$

Taking (116) and (117) together we obtain

$$\partial_s v^\mu = F_\infty^{\mu\nu} (z(s)) v_\nu + \frac{4e^2}{3} \vartheta [\ddot{v}^\mu + v^\mu (\dot{v}\dot{v})]$$

implying (7).

#### 4. MANY INTERACTING CHARGES

When we consider a single charge the EM field  $A$  seems quite similar to the adjacent field  $A^{\text{ad}}$ . To see the difference we have to consider many interacting charges, and from this construction one can see that the EM field potential  $A$  is responsible for the interaction between the charges and it is an observable field, whereas  $A^{\text{ad}}$  is solely responsible for an internal interaction of a charge with itself.

We consider a closed system of  $N$  charges with densities  $\psi_\ell$ ,  $\ell = 1, \dots, N$ ; the charges interact through the electromagnetic field  $A = A^\mu$  and every charge interacts also with its adjacent field  $A_\ell^{\text{ad}}$ , and external EM field is absent  $A_{\text{ex}}^\mu = 0$ .

The Lagrangian for the system of interacting charges has the form

$$(118) \quad \mathcal{L} = \sum_\ell L_{\text{KG}} (\psi_\ell, \psi_{\ell;\mu}, \psi_\ell^*, \psi_{\ell;\mu}^*) - L_{\text{em}} (A) + \sum_\ell L_{\text{em}} (A_\ell^{\text{ad}}).$$

It involves covariant derivatives

$$(119) \quad \psi_\ell^\mu = \tilde{\partial}_\ell^\mu \psi_\ell = \partial^\mu \psi_\ell + \frac{iq_\ell}{\hbar} \tilde{A}_\ell^\mu \psi_\ell,$$

$$(120) \quad \tilde{\partial}_\ell^\mu = \partial^\mu + \frac{iq_\ell}{\hbar} \tilde{A}_\ell^\mu, \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_t, -\nabla), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla)$$

$$(121) \quad \tilde{A}_\ell^\mu = A^\mu - A_\ell^{\text{ad}\mu}$$

Here  $\psi_\ell^*$  is complex conjugate to  $\psi_\ell$ , and the speed of light  $c = 1$ . In the simplest case which we consider here  $\psi_\ell$  is a complex scalar and  $L_{\text{KG}}$  is the Lagrangian (12) of nonlinear Klein-Gordon (KG) equation. The field equations which describe dynamics of the system are obtained as Euler-Lagrange equations for the Lagrangian  $\mathcal{L}$  and involve nonlinear Klein-Gordon (KG) equations for the charge distributions  $\psi_\ell$  and Maxwell equations for the potentials  $A, A_\ell^{\text{ad}}$ . The KG equations have the form

$$(122) \quad \left[ \tilde{\partial}_\ell^\mu \tilde{\partial}_\ell^\mu + \kappa_0^2 + G'(\psi_\ell^* \psi_\ell) \right] \psi_\ell = 0$$

$G'(s) = dG/ds$ . Equations for the fields take the form of Maxwell equations for the fields  $F^{\mu\nu}, F_\ell^{\text{ad}\mu\nu}$

$$(123) \quad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu, \quad J^\nu = \sum_\ell J_\ell^\nu$$

$$(124) \quad \partial_\mu F_\ell^{\text{ad}\mu\nu} = 4\pi J_\ell^\nu,$$

where the source currents are defined by

$$(125) \quad J_\ell^\nu = -\frac{\partial L_\ell}{\partial A_\nu} = -i\frac{q_\ell}{\hbar} \left( \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}} \psi_\ell - \frac{\partial L_\ell}{\partial \psi_{\ell;\nu}^*} \psi_\ell^* \right).$$

Every current  $J_\ell^\nu$  satisfies the continuity equation, therefore we impose Lorentz gauge on solutions of (123), (124). We choose solution operator so that solution of (123) is given by (3), and similarly to (4)  $A_\ell^{\text{ad}\nu} = \vartheta_\ell \mathcal{G}_{\text{ret}} J^\nu + (1 - \vartheta_\ell) \mathcal{G}_{\text{adv}} J^\nu$ . Since the Maxwell equation is linear, the total EM field is a superposition of fields for a single charge:

$$A^\mu = \sum_\ell A_\ell^\mu, \quad F^{\mu\nu} = \sum_\ell F_\ell^{\mu\nu}$$

with the following equation for every field:

$$(126) \quad \partial_\mu F_\ell^{\mu\nu} = 4\pi J_\ell^\nu.$$

It is important to note that KG equation for  $\ell$ -th charge involves (through covariant derivative) the electromagnetic potential

$$(127) \quad \tilde{A}_\ell^\mu = A_{\neq \ell}^\mu + A_\ell^\mu - A_\ell^{\text{ad}\mu}$$

where

$$A_{\neq \ell}^\mu = \sum_{\ell' \neq \ell} A_{\ell'}^\mu.$$

Comparing with (15) we see that for  $\ell$ -th charge the potential  $A_{\neq \ell}^\mu$  plays the role of an external field  $A_{\text{ex}}^\mu$  for this charge. If we add a test charge to the system, it becomes  $N+1$ -st charge and the field potential  $A_{\neq N+1}^\mu$  coincides with the total field  $A^\mu$  determined from (123). Therefore the measurable field is the field  $F^{\mu\nu}$  with the potential  $A^\mu$ . Note that none of the adjacent potentials  $A_\ell^{\text{ad}\mu}$ ,  $\ell = 1, \dots, N$ , enters the equations which determine the dynamics of the test charge. Therefore  $A_\ell^{\text{ad}\mu}$  describes solely an internal self-interaction of a charge and this property completely differs the adjacent field from the EM field which acts on all charges of the system.

## 5. APPENDIX: ELEMENTARY SURFACE INTEGRALS

The integral of the form (105) determines a bilinear form on linear functions  $\mathbf{f}$  on 3D space:

$$(128) \quad \Xi(\mathbf{f}_1, \mathbf{f}_2) = \int_{|\mathbf{x}|=1} \frac{\mathbf{f}_1 \mathbf{f}_2}{(R^2 + (\mathbf{x}\mathbf{v})^2)^{3/2}} d\sigma$$

As a first step we evaluate the form for special cases. If we take the axis  $x_3$  axis along  $\mathbf{v}$  we obtain  $\mathbf{v} = (0, 0, v^3)$ ,  $|\mathbf{v}| = |v_3|$  and we consider

$$\int x_i x_j \epsilon^{-3} d\sigma = \int \frac{x_i x_j}{(R^2 + (x_3^2 v_3^2))^{3/2}} d\sigma = \int \frac{x_i x_j}{(R^2 + (x_3^2 v_3^2))^{3/2}} d\sigma.$$

Since  $x_i x_j$  is odd with respect to  $x_i$  for  $i \neq j$  we obtain

$$(129) \quad \frac{1}{R} \int x_i x_j \epsilon^{-3} d\sigma = 0 \quad \text{for } i \neq j$$

In spherical coordinates  $x_3 = r \cos \theta, x_1 = r \sin \theta \cos \varphi, x_2 = r \sin \theta \sin \varphi$

$$\Xi(x_i, x_j) = R^{-1} \int_{-\pi}^{\pi} \int_0^{\pi} \frac{x_i x_j}{(R^2 + x_3^2 v_3^2)^{3/2}} r^2 \sin \theta d\theta d\varphi$$

The integral

$$(130) \quad \begin{aligned} \frac{1}{R} \int_{-\pi}^{\pi} \int_0^{\pi} \frac{x_3^2}{(R^2 + x_3^2 v_3^2)^{3/2}} r^2 \sin \theta d\theta d\varphi &= \int_{-\pi}^{\pi} \int_0^{\pi} \frac{\cos^2 \theta}{(1 + v_3^2 \cos^2 \theta)^{3/2}} \sin \theta d\theta d\varphi \\ &= 4\pi \int_0^1 \frac{t^2}{(1 + t^2 v_3^2)^{3/2}} dt = \Xi_{\parallel}(|v_3|) \end{aligned}$$

where

$$(131) \quad \Xi_{\parallel}(b) = \frac{4\pi}{b^3} \int_0^b \frac{t^2}{(1 + t^2)^{3/2}} dt = -\frac{4\pi}{b^3} \frac{b - (\ln(b + \sqrt{b^2 + 1})) \sqrt{b^2 + 1}}{\sqrt{b^2 + 1}}$$

Therefore

$$(132) \quad \Xi\left(\frac{\mathbf{v}}{|\mathbf{v}|}, \frac{\mathbf{v}}{|\mathbf{v}|}\right) = \int_{\Omega} x_3^2 \epsilon^{-3} d\sigma = \Xi_{\parallel}(|\mathbf{v}|)$$

For the orthogonal to  $\mathbf{v}$  direction

$$\frac{1}{R} \int_{-\pi}^{\pi} \int_0^{\pi} \frac{x_1^2}{(R^2 + x_3^2 v_3^2)^{3/2}} r^2 \sin \theta d\theta d\varphi = 2\pi \int_0^1 \frac{1 - t^2}{(1 + t^2 v_3^2)^{3/2}} dt = \Xi_{\perp}(|v_3|)$$

where

$$(133) \quad \Xi_{\perp} = 2\pi \frac{1}{\sqrt{|\mathbf{v}|^2 + 1}} - \frac{1}{2} \Xi_{\parallel}(|\mathbf{v}|)$$

Therefore for  $\mathbf{f}$  orthogonal to  $\mathbf{v}$  such that  $\mathbf{f} \cdot \mathbf{v} = 0$  and  $|\mathbf{f}| = 1$

$$\Xi(\mathbf{f}, \mathbf{f}) = \Xi_{\perp}(|\mathbf{v}|)$$

Now we consider a general functional  $\mathbf{f}$  on  $\mathbb{R}^3$ , the functional can be written in the form  $\mathbf{f} = f_i x_i$  and we define its norm  $|\mathbf{f}| = (f_i f_i)^{1/2}$ . The functional can be expanded as

$$\mathbf{f} = \frac{1}{\mathbf{v}^2} (\mathbf{f} \cdot \mathbf{v}) \mathbf{v} + \mathbf{f}_{\perp}, \quad \mathbf{f}_{\perp} = \frac{1}{\mathbf{v}^2} (\mathbf{v}^2 \mathbf{f} - \mathbf{v} (\mathbf{f} \cdot \mathbf{v})), \quad \mathbf{f}_{\perp} \cdot \mathbf{v} = 0.$$

We can turn the axis so that  $\mathbf{v}$  is along  $x_3$  and  $\mathbf{f}_{\perp}$  is along  $x_1$ , the sphere is preserved, the functional becomes proportional to  $x_3$  and the norm of the functionals  $\mathbf{f}_{\perp}$  and  $\mathbf{f} - \mathbf{f}_{\perp}$  are preserved. According to (129)  $\Xi(\mathbf{v}, \mathbf{f}_{\perp}) = 0$ , therefore

$$\Xi(\mathbf{f}, \mathbf{f}) = \Xi(\mathbf{f} - \mathbf{f}_{\perp}, \mathbf{f} - \mathbf{f}_{\perp}) + \Xi(\mathbf{f}_{\perp}, \mathbf{f}_{\perp})$$

and the quadratic form takes the form

$$\Xi(\mathbf{f}, \mathbf{f}) = \frac{1}{\mathbf{v}^2} (\mathbf{f} \cdot \mathbf{v})^2 \Xi\left(\frac{\mathbf{v}}{|\mathbf{v}|}, \frac{\mathbf{v}}{|\mathbf{v}|}\right) + |\mathbf{f}_{\perp}|^2 \Xi\left(\frac{\mathbf{f}_{\perp}}{|\mathbf{f}_{\perp}|}, \frac{\mathbf{f}_{\perp}}{|\mathbf{f}_{\perp}|}\right)$$

Therefore

$$(134) \quad \Xi(\mathbf{f}, \mathbf{f}) = \frac{1}{\mathbf{v}^2} (\mathbf{f} \cdot \mathbf{v})^2 \Xi_{\parallel} + |\mathbf{f}_{\perp}|^2 \Xi_{\perp}.$$

In particular, for a functional  $\mathbf{f} = x_i$

$$(135) \quad \Xi(x_i, x_i) = \frac{1}{\mathbf{v}^2} v_i^2 \Xi_{\parallel} + \left(1 - \frac{1}{\mathbf{v}^2} v_i^2\right) \Xi_{\perp}$$

For  $i \neq j$   $|(x_i + x_j)|^2 = |(1, 1, 0)|^2 = 2$  and

$$(136) \quad \Xi(x_i + x_j, x_i + x_j) = \frac{1}{\mathbf{v}^2} (v_i + v_j)^2 \Xi_{\parallel} + \left(2 - \frac{1}{\mathbf{v}^2} (v_i + v_j)^2\right) \Xi_{\perp}$$

Therefore, for  $i \neq j$

$$(137) \quad \begin{aligned} \Xi(x_i, x_j) &= \frac{1}{4} (\Xi(x_i + x_j, x_i + x_j) - \Xi(x_i - x_j, x_i - x_j)) \\ &= \frac{1}{4\mathbf{v}^2} \left[ \left( (v_i + v_j)^2 - (v_i - v_j)^2 \right) \Xi_{\parallel} + \left( (v_i - v_j)^2 - (v_i + v_j)^2 \right) \Xi_{\perp} \right] = \frac{v_i v_j}{\mathbf{v}^2} [\Xi_{\parallel} - \Xi_{\perp}] \end{aligned}$$

$$(137) \quad \Xi(x_i, x_j) = \frac{v_i v_j}{\mathbf{v}^2} [\Xi_{\parallel} - \Xi_{\perp}] \quad \text{for } i \neq j.$$

Since (135) can be written in the form

$$(138) \quad \Xi(x_i, x_i) = \frac{v_i^2}{\mathbf{v}^2} [\Xi_{\parallel} (|\mathbf{v}|) - \Xi_{\perp} (|\mathbf{v}|)] + \Xi_{\perp} (|\mathbf{v}|)$$

we obtain expression (106) for the general case.

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